Probability of Rolling Heavy Bombers

INTRODUCTION

In Axis and Allies, it can be very advantageous to possess the heavy bombers technology. But it can also be expensive. This paper examines the exact probability of achieving the heavy bomber technology, given various conditions.

No Techs, One Die

Suppose the player has no techs and purchases one roll. The probability of developing a tech is one in six. The probability of that tech being heavy bombers is one in six. The intersection is one in thirty six.

No Techs, Two Dice

The probability of rolling one tech with two dice is one in six times five in six times two choose 1 (the probability of one die rolling six and one die not rolling six). The probability of that tech being heavy bombers is the total probability (1) minus the chance of NOT rolling heavy bombers, which is five in six. That last sentence might seem like a round about way of saying one in six, which is true, but will make more sense later. $\frac{1}{6}\frac{5}{6}\binom{2}{1}(1-\frac{5}{6})$

The probability of rolling two techs with two dice is one over six squared time two choose two. The probability that one of those techs is heavy bombers is the total probability (1) minus the probability that the first roll isn't heavy bombers (5 out of six) times the probability that the second roll also isn't heavy bombers (4 out of 5). $\frac{1}{6}\frac{1}{6}\binom{2}{2}(1-\frac{5}{6}\frac{4}{5})$

So the total probability of rolling heavy bombers with two dice, if no techs are already owned, is:

$$P_{02}=\frac{1}{6}\frac{5}{6}\binom{2}{1}(1-\frac{5}{6})+\frac{1}{6}\frac{1}{6}\binom{2}{2}(1-\frac{5}{6}\frac{4}{5})$$

No techs, 1 to 5 Dice

It isn't hard to generalize the following rule:

$$P_{0n}=\sum_{i=1}^{n}\left(\frac{1}{6}\right)^{i}\left(\frac{5}{6}\right)^{n-i}\binom{n}{i}\left(1-\frac{5!}{(5-i)!}\frac{(6-i)!}{6!}\right)\qquad 1\leqslant n\leqslant 5$$

Which works well up to five tech dice.

I like to get whole number ratios (avoiding round off error in large sums), so I multiply both sides by 6^n 6!:

$$P_{0n}(6^{n})6!=\sum_{i=1}^{n}5^{n-i}\binom{n}{i}\left(6!-\frac{5!}{(5-i)!}(6-i)!\right)\qquad 1\leqslant n\leqslant 5$$

Formula #1

No techs, 6 Dice

At six dice the formula given above doesn't work. Because if all six dice roll sixes (very unlikely, but it could happen) then the player gets all six techs, so in that unusual occurrence heavy bombers is guaranteed. The probability of rolling six sixes is 1 in six to the exponent six. So we modify the previous formula like this (where n=6):

$$P_{0n}=\sum_{i=1}^{5}\left(\frac{1}{6}\right)^{i}\left(\frac{5}{6}\right)^{n-i}\binom{n}{i}\left(1-\frac{5!}{(5-i)!}\frac{(6-i)!}{6!}\right)+\left(\frac{1}{6}\right)^{6}\qquad n=6$$

No techs, six or more dice

And once again, it is not difficult to generalize this result for n>6:

$$P_{0n}=\sum_{i=1}^{5}\left(\frac{1}{6}\right)^{i}\left(\frac{5}{6}\right)^{n-i}\binom{n}{i}\left(1-\frac{5!}{(5-i)!}\frac{(6-i)!}{6!}\right)+\sum_{i=6}^{n}\left(\frac{1}{6}\right)^{i}\left(\frac{5}{6}\right)^{n-i}\binom{n}{i}\qquad n\geqslant 6$$

And since I like to have whole number ratios:

$$P_{0n}\times(6^{n})6!=\sum_{i=1}^{5}5^{n-i}\binom{n}{i}\left(6!-\frac{5!}{(5-i)!}(6-i)!\right)+6!\sum_{i=6}^{n}5^{n-i}\binom{n}{i}\qquad n\geqslant 6$$

Formula #2

One tech

Suppose that the player already has one tech (not heavy bombers). The player buys one tech roll. The chance of rolling a six is one in six. The chance of rolling heavy bombers is now 1 in 5, or 1 minus 4 in 5. P_{11}=1 in 30.

The logic for multiple dice is similar to the logic where the player has no techs, except now the odds of rolling heavy bombers have improved:

$$P_{1n} = \sum_{i=1}^{n} \left(\frac{1}{6}\right)^{i} \left(\frac{5}{6}\right)^{n-i} \binom{n}{i} \left(1 - \frac{4!}{(4-i)!} \frac{(5-i)!}{5!}\right) \qquad 1 \leqslant n \leqslant 4$$

And to make ratios of whole numbers I multiply both sides by (6^n) 5!

$$P_{1n} 6^{n} 5! = \sum_{i=1}^{n} 5^{n-i} \binom{n}{i} \left(5! - \frac{4!}{(4-i)!} (5-i)!\right) \qquad 1 \leqslant n \leqslant 4 \quad \textbf{Formula \#3}$$

And for more than 5

$$P_{1n} = \sum_{i=1}^{4} \left(\frac{1}{6}\right)^{i} \left(\frac{5}{6}\right)^{n-i} \binom{n}{i} \left(1 - \frac{4!}{(4-i)!} \frac{(5-i)!}{5!}\right) + \sum_{i=5}^{n} \left(\frac{1}{6}\right)^{i} \left(\frac{5}{6}\right)^{n-i} \binom{n}{i} \qquad n \geqslant 5$$

Fixed for whole numbers

$$P_{1n} 6^{n} 5! = \sum_{i=1}^{4} 5^{n-i} \binom{n}{i} \left(5! - \frac{4!}{(4-i)!} (5-i)!\right) + 5! \sum_{i=5}^{n} 5^{n-i} \binom{n}{i} \qquad n \geqslant 5 \quad \textbf{Formula \#4}$$

Two to Five Techs

I think the pattern should be rather obvious by now:

$$P_{2n} 6^{n} 4! = \sum_{i=1}^{n} 5^{n-i} \binom{n}{i} \left(4! - \frac{3!}{(3-i)!} (4-i)!\right) \qquad 1 \leqslant n \leqslant 3 \quad \textbf{Formula \#5}$$

$$P_{2n} 6^{n} 4! = \sum_{i=1}^{3} 5^{n-i} \binom{n}{i} \left(4! - \frac{3!}{(3-i)!} (4-i)!\right) + 4! \sum_{i=4}^{n} 5^{n-i} \binom{n}{i} \qquad n \geqslant 4 \quad \textbf{Formula \#6}$$

$$P_{3n} 6^{n} 3! = \sum_{i=1}^{n} 5^{n-i} \binom{n}{i} \left(3! - \frac{2!}{(2-i)!} (3-i)!\right) \qquad 1 \leqslant n \leqslant 2 \quad \textbf{Formula \#7}$$

$$P_{3n} 6^{n} 3! = \sum_{i=1}^{2} 5^{n-i} \binom{n}{i} \left(3! - \frac{2!}{(2-i)!} (3-i)!\right) + 3! \sum_{i=3}^{n} 5^{n-i} \binom{n}{i} \qquad n \geqslant 3 \quad \textbf{Formula \#8}$$

$$P_{4n}(6)(2) = 1 \qquad n = 1 \quad \textbf{Formula \#9}$$

$$P_{4n} 6^{n} 2 = 5^{n-1} \binom{n}{1} + 2 \sum_{i=2}^{n} 5^{n-i} \binom{n}{i} \qquad n \geqslant 2 \quad \textbf{Formula \#10}$$

$$P_{5n} 6^{n} = \sum_{i=4}^{n} 5^{n-i} \binom{n}{i} \qquad n \geqslant 1 \quad \textbf{Formula \#11}$$

The last three formulae look a bit different.

Tables, Percentage Chance of Getting Heavy Bombers

Number of Dice	***No Techs***	***One Tech***	***Two Techs***	***Three Techs***	***Four Techs***	***Five Techs***
1	2.78	3.33	4.17	5.56	8.33	16.67
2	5.56	6.67	8.33	11.11	16.67	30.56
3	8.33	10.00	12.50	16.67	24.77	42.13
4	11.11	13.33	16.67	22.20	32.48	51.77
5	13.89	16.67	20.83	27.66	39.72	59.81
6	16.67	20.00	24.98	33.02	46.42	66.51
7	19.44	23.33	29.11	38.23	52.56	72.09
8	22.22	26.66	33.21	43.25	58.14	76.74
9	25.00	29.98	37.25	48.06	63.18	80.62
10	27.77	33.28	41.21	52.63	67.70	83.85
11	30.54	36.56				
12	33.31	39.81				
13	36.06	43.02				
14	38.81					
15	41.53					